NUMERICAL COMPUTATION OF FULLY-DEVELOPED TURBULENT CHANNEL FLOW BASED ON THE STREAMFUNCTION-VORTICITY FORMULATION

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Abstract
The Czibere’s new stochastical turbulence model is an extension of the Kármán’s two-dimensional similarity hypothesis [1] to three-dimensional case [2,3]. The author considers the numerical computation of fully-developed channel flow based on the new vortex theorem of Czibere’s new stochastical turbulence model [2,3]. A streamfunction equation and an algebraic equation for the turbulent stresses are added to the numerical model. The partial differential equation system for the streamfunction and vorticity has been solved by implicit methods with Dirichlet and Neumann boundary conditions. The numerical results are compared with the analytical solution by G. Janiga [5].

Key words: turbulent channel flow, turbulence-models, streamfunction-vorticity formulation

INTRODUCTION

Tibor Czibere (2000) [2,3] extended Kármán’s two-dimensional similarity hypothesis [1] which was born in the first half of the 20th century to the three-dimensional case, because the turbulence phenomena is always three-dimensional. The Czibere’s new stochastical turbulence model and the theoretical background of the subsequent numerical computation are introduced. The author deals with the basic equations of fully-developed turbulent channel flow, using the new stochastical turbulence model, and describes the numerical solution of the turbulent flow task. The results are compared at different Reynolds numbers with the analytical solution by G. Janiga [5]. The results and conclusions are summarized in the last section.

THEORETICAL BACKGROUND

In the Czibere’s new stochastical turbulence model a natural coordinate system \( q'_1, q'_2, q'_3 \) is attached to the mean velocity field of the flow, for which the base vectors are determined by \( \mathbf{v} \) and rot \( \mathbf{v} \) vectors. The second coordinate vector is determined as \( e'_2 = \mathbf{v} \times \text{rot} \mathbf{v} \), and the third as the negative of \( \text{rot} \mathbf{v} \) vector. The first coordinate vector is given by the vectorial product of the previous two ones, that is \( e'_1 = e'_2 \times e'_3 \). There exists a relation between the turbulent vorticity and the dominant shear stress near the wall in the natural orthogonal coordinate system. According to Czibere’s new stochastical turbulence model the Reynolds’ stress tensor can be written as [2]:

\[
F_R = \rho x^2 l^2 H |\Omega|\Omega, \tag{1}
\]
where $\rho$ is the density of the fluid, $\kappa = 0.40704$ is the Kármán constant, $l$ is an appropriately chosen scale function of the turbulence model. The $H$ is the so-called similarity tensor of turbulence based on the theory of T. Czikere [2]. The $\Omega$ vorticity determines the magnitude of the $\nabla \times \mathbf{v}$ vector.

Let us introduce the notation for the part of the Reynolds' stress tensor, which is not containing the $H$ similarity tensor as follows:

$$\Theta(q_1', q_2', q_3', t) = \rho \kappa^2 l^2 |\Omega| \Omega,$$

which is the second element of first row and first column of the Reynolds' stress tensor and this is the so-called dominant turbulent stress in the shear flow.

During the computations, it is difficult to use the natural coordinate system determined by the velocity vectors, because the directions of the coordinate system change from point to point, and can change with time at a given point as well. Therefore it is convenient to use a general orthogonal curvilinear coordinate system $q_1, q_2, q_3$, which is independent of the velocity vectors and represents the given flow problem. The Eq. (1) Reynolds’ stress tensor in the computational coordinate system $q_1, q_2, q_3$ is [2]:

$$\mathbf{F}_R = \Theta(q_1, q_2, q_3) \mathbf{G},$$

where the tensor $\mathbf{G}$ is similarity tensor $H$ of the natural coordinate system transformed into the computational coordinate system.

Now, we consider the incompressible viscous flow in potential field of force $(\mathbf{g} = -\nabla U)$, in which the fluid particles can have turbulent motion. In the orthogonal curvilinear coordinate system $q_1, q_2, q_3$ the mean velocity is $\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3$, and the turbulent velocity fluctuation is $\mathbf{v}' = v'_1 \mathbf{e}_1 + v'_2 \mathbf{e}_2 + v'_3 \mathbf{e}_3$.

The conservation of mass is expressed by the continuity equation in the mean velocity field:

$$\nabla \cdot \mathbf{v} = 0,$$

and in the velocity field of turbulent fluctuation:

$$\nabla \cdot \mathbf{v}' = 0,$$

and the turbulent motion of the viscous fluid is described by the Reynolds’ momentum equation. Czikere derives new vortex theorems to the analogy of Helmholtz based on the Friedmann conservation law of vector tubes [2] in case of constant density of the fluid. We may utilize the vortex theorem originated from the momentum equation by the curl operation in case of numerical computation of viscous flow, which applies to the conservation characteristics of vortex lines determined by the equation $\text{rot} \, \mathbf{v} \times d\mathbf{r} = \mathbf{0}$ [2]:

$$\frac{\partial \Omega}{\partial t} + (\mathbf{v} \cdot \nabla) \Omega - (\mathbf{\Omega} \cdot \nabla) \mathbf{v} = \nu \Delta \mathbf{\Omega} + \frac{1}{\rho} \nabla \times \text{Div}(\Theta \mathbf{G}^*) .$$

where $\nu$ is the kinematic viscosity, and $\mathbf{G}^*$ is the transformation of the deviator $H^*$ of the natural coordinate system into the computational coordinate system. According to the partial differential equation of Czikere’s new vortex diffusion in the velocity field of turbulent fluctuation, the
relationship between the velocity field of the turbulent fluctuation and the vorticity of the mean velocity is clearly indicated [2].

**THE MATHEMATICAL MODEL OF THE FLOW**

The further analysis will be performed in a computational rectangular coordinate system. The continuity equation Eq. (4) in case of turbulent channel flow:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{7}
\]

The component of the vorticity vector:

\[
\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{8}
\]

Based on the continuity equation Eq. (7) the streamfunction is introduced, thus the velocity components are:

\[
u = \frac{\partial \Psi}{\partial y}; \quad \nu = -\frac{\partial \Psi}{\partial x}.
\]

Substituting the obtained velocity components into Eq. (8), we arrive at the following Poisson equation for the streamfunction:

\[
\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = -\Omega. \tag{9}
\]

Both velocity components are known at the inlet boundary, thus both \(u\) and \(v\) are specified for the streamfunction values. Since the solid surface is stationary and hence the two velocity components are also zero, that is the both derivatives of the streamfunction \(\Psi\) are given. In the outflow boundary condition the normal derivatives of both velocity components are zero, therefore the normal derivatives of the streamfunction \(\Psi\) are also zero.

The scalar differential equation of Czibere's new vorticity transport equation for the turbulent flow of the incompressible viscid fluid in the mean velocity field applies:

\[
\frac{\partial \Omega}{\partial t} + u \frac{\partial \Omega}{\partial x} + v \frac{\partial \Omega}{\partial y} = \nu \left( \frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} \right) + \\
\frac{1}{\rho} \left[ \frac{\partial \left( \Theta G^*_1 \right)}{\partial x} + \frac{\partial \left( \Theta G^*_2 \right)}{\partial y} \right] - \frac{\partial \left( \Theta G^*_1 \right)}{\partial x} - \frac{\partial \left( \Theta G^*_2 \right)}{\partial y} \right]. \tag{10}
\]

where the elements of tensor \(G^*_i\) can be expressed by the velocity components, and \(\sqrt{u^2 + v^2}\) is the absolute value of the mean velocity:
\[ G_{11}^* = \alpha_* \left( \frac{v}{v'} \right)^2 + \beta_* \left( \frac{u}{v'} \right)^2 + 2 \frac{uv}{v'^2} \frac{\Omega}{\Omega'}^2 + G_{12}^* = G_{21}^* = (\alpha_* - \beta_*) \frac{vu}{v'^2} \frac{\Omega}{\Omega'} \left( \frac{v}{v'} \right)^2 \left( \frac{u}{v'} \right)^2; \]

\[ G_{22}^* = \alpha_* \left( \frac{u}{v'} \right)^2 + \beta_* \left( \frac{v}{v'} \right)^2 - 2 \frac{uv}{v'^2} \frac{\Omega}{\Omega'}; \quad G_{33}^* = \gamma_* . \]

The constants \( \alpha_*, \beta_*, \gamma_* \) are the diagonal elements of the \( \mathbf{H}^* \) deviator of Czibere’s new stochastical turbulence model: \( \alpha_* = -1.055 \); \( \beta_* = 1.055 \); and \( \gamma_* = 0 \).

The boundary conditions for vorticity is constructed from the values of streamfunction in case of the inlet and the solid wall. The normal derivative of the component of the vorticity vector is also zero at the outflow.

The algebraic equation suitable to calculate the \( \Theta(x, y, t) \) turbulent dominant stress in the computational coordinate system is:

\[ \Theta(x, y, t) = \rho k^{2} l^{2} \Omega^{2} . \]  

(11)

The unknown functions \( \psi(x, y, t) \), \( \Omega(x, y, t) \), \( \Theta(x, y, t) \) can be determined with the help of the Eqs. (9) to (11) by iterations.

NUMERICAL SOLUTION

The Eq. (9) elliptical partial differential equation of the streamfunction \( \psi(x, y, t) \) is discretized applying central finite difference approximations on a rectangular domain. During the solution, a non-equidistant partitioning is assumed, and the solution is found by an implicit method.

Utilizing central finite differences, from the differential equation Eq. (9) the following difference equation is obtained in the \( (i,j) \) index grid:

\[ L_{xx} \psi_{i,j} + L_{yy} \psi_{i,j} = - \Omega_{i,j} , \]

(12)

where \( L_{xx} \) and \( L_{yy} \) are second-order difference-operators in axial and radial directions.

The obtained five banded linear equation system is solved by the Nyiri’s new direct method for solving band linear system [4].

The basic form of Eq. (10) Czibere’s new turbulent vorticity transport equation is obtained applying ADI (Alternating Direction Implicit) method by Douglas-Gunn-algorithm extended to an inhomogeneous difference equation:

\[ \frac{\Delta \Omega_{i,j}^{n+1}}{\Delta t} - (1 - \omega) \left[ L_{xx} \Omega_{i,j}^{n} + L_{yy} \Omega_{i,j}^{n} - u_{i,j} L_{x} \Omega_{i,j}^{n} - v_{i,j} L_{y} \Omega_{i,j}^{n} \right] - \]

\[ - \omega \left[ L_{xx} \Omega_{i,j}^{n+1} + L_{yy} \Omega_{i,j}^{n+1} - u_{i,j} L_{x} \Omega_{i,j}^{n+1} - v_{i,j} L_{y} \Omega_{i,j}^{n+1} \right] = \frac{\bar{T}_{i,j}^n}{\rho} , \]

(13)
where \( L_x \) and \( L_y \) are first-order difference-operators in axial and radial directions, \( \omega \) is the relaxation parameter, and \( \Delta \Omega_{i,j}^{n+1} = \Omega_{i,j}^{n+1} - \Omega_{i,j}^{n} \) is the difference of the vorticity distribution between two time-levels in the discrete points of the rectangular domain, and

\[
T_{i,j}^{n+1/2} = L_{xx} \left( \Theta G_{21}^* \right)_{i,j} + L_{xy} \left( \Theta G_{22}^* \right)_{i,j} - L_{yx} \left( \Theta G_{11}^* \right)_{i,j} - L_{yy} \left( \Theta G_{12}^* \right)_{i,j},
\]

is the turbulent source term of the right hand side of the Eq. (13) difference equation at the \( n+(1/2) \) time-step of the iteration. The \( \omega \) relaxation parameter can be chosen from the following form: \((0 \leq \omega \leq 1)\). At first, the forward time centered space (FTCS) scheme is obtained in case of \( \omega = 0 \). If \( \omega = 0.5 \) the Crank-Nicholson scheme is obtained and if \( \omega = 1.0 \) the fully implicit scheme is given. A von Neumann stability analysis of Eq. (13) indicates that the solution is stable applying the Crank-Nicholson scheme, therefore we choose the relaxation parameter \( \omega = 0.5 \). During the ADI–method a linear equation system with \( x \) and \( y \) direction tridiagonal coefficient matrix must be solved, whose solution can be easily obtained with the help of the well-known, efficient and popular Thomas-algorithm.

The numerically computed dimension-less velocity profiles can be seen at different Reynolds numbers in Figures 1. to 2. The Reynolds number is defined by the value of average velocity, and \( V_m \) is the maximum value of the velocity.

The new vortex theorem of Czibere’s new stochastical turbulence model is efficient suitable for the computation of the fully-developed turbulent channel flow. On the basis of Czibere’s three-dimensional stochastical turbulence model, described above, a tensor equation can be given to the Reynolds’ stress tensor and its deviator, in which there is only one unknown scalar function, the so-called dominant turbulent shear stress.

![Fig. 1 Comparison of velocity profiles at Re=23300](image_url)
Fig. 2 Comparison of velocity profiles at $Re=105000$

With the help of Czibere’s tensor equation of Reynolds’ stress tensor and its deviator, a closed differential equation system can be set up for numerical computation of the turbulent boundary layer flow of viscid fluid.

REFERENCES