Approximating the helix with rational cubic Bézier curves

Imre Juhász

The paper is on the approximation of a cylindrical helix by cubic rational Bézier curves. Two methods are given, one which guarantees exact slopes at the end-points of the approximating curve, and another which ensures 2nd-order continuity at the junction of two approximating curves. In both cases, there are error bounds which enable the helix to be approximated within any prescribed tolerance.

Keywords: rational polynomials, Bézier curves, helices

Because of their flexibility, the most currently widespread curve and surface descriptions are rational curves and surfaces, especially NURBS. Increasing numbers of CAD systems and graphics libraries offer this method of curve and surface definition. That is why it is meaningful to deal with the approximation of known curves and surfaces by rational ones, even if the former can be described by exact mathematical formulae.

The objective of this paper is to approximate a cylindrical helix by cubic rational Bézier curves. The helix is a unique curve, since its curvature and torsion are constant; the helix is the only spatial curve with this property. There are only two plane curves which share this property; the circle and the straight line. A practical consequence of this property is that the helix can be moved onto itself, i.e. by applying the appropriate helical motion to any arc of the curve, the moving arc remains on the original helix at any of its positions. This property can be utilized in the approximation of a helix, since, if we can approximate any arc of the helix, this approximating arc can be used repeatedly. Another important characteristic of our approximation is the symmetry of the approximating curves, which is also due to the constant curvature and torsion of the helix. Although the helix is a special curve with advantageous properties, it is an irrational curve. Note that a straight line can have an infinite number of points of intersection with a helix. Because of this, there is no exact rational representation of the helix, i.e. it is meaningful to investigate rational approximations.

In an earlier paper, a helix was approximated by cubic and quartic rational Bézier curves, following a kinematic approach, but without error calculations. In this paper, we propose two rather elementary methods for the solution of the problem above, and we include error calculations. For the sake of simplicity, we consider a helix with a parameter \( p \) which is on a circular cylinder of radius \( r \) and axis \( z \). This is not a restriction, because one can always achieve such a situation by applying an appropriate coordinate system transformation. The parametric form of the helix under consideration is

\[
h(v) = (r \cos(v), r \sin(v), pv) \quad v \in \mathbb{R}
\]

RATIONAL BÉZIER CURVE

We will use the parametric form of rational Bézier curves, which is

\[
r(t) = \sum_{i=0}^{n} B_i^n(t) w_i b_i \quad t \in [0,1]
\]

where the \( b_i \) are control points, \( 0 < w_i \in \mathbb{R} \) are weights, and

\[
B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}
\]

are the Bernstein polynomials of degree \( n \), \( 0 \leq i \leq n \).

We are going to examine the cubic case, i.e. \( n = 3 \), where

\[
r(t) = \frac{w_0 b_0 B_0^3(t) + w_1 b_1 B_1^3(t) + w_2 b_2 B_2^3(t) + w_3 b_3 B_3^3(t)}{w_0 B_0^3(t) + w_1 B_1^3(t) + w_2 B_2^3(t) + w_3 B_3^3(t)}
\]

The first and second derivatives at the end-points will also be needed. These are

\[
\mathbf{r}(0) = \frac{3w_1}{w_0} (b_1 - b_0)
\]

\[
\mathbf{r}(1) = \frac{3w_2}{w_3} (b_3 - b_2)
\]
Approximating the helix with rational cubic Bézier curves: I Juhász

\[
\begin{align*}
\mathbf{r}(0) &= \frac{6(w_0 w_1 - 3w_2^2)}{w_0^2}(b_1 - b_0) + \frac{6w_0}{w_0}(b_2 - b_0) \\
\mathbf{r}(1) &= \frac{6(3w_2^2 - w_3 w_3)}{w_3^2}(b_3 - b_2) - \frac{6w_1}{w_3}(b_3 - b_1)
\end{align*}
\]  

(3)

**BASIC IDEA OF APPROXIMATION**

We approximate an arc of the helix. The approximation process consists of the following steps.

1. Describe a circular arc in the \((x, y)\) coordinate plane as a rational Bézier curve of degree 2.
2. Elevate the degree of the previous rational Bézier curve by 1.
3. Lift out the control points of the cubic rational Bézier curve from the \((x, y)\) coordinate plane along a line parallel to the \(z\) axis.

The process described above results in a cubic rational Bézier curve on the cylinder of the helix to be approximated.

A circular arc on the \((x, y)\) coordinate plane with radius \(r\), central angle \(2\alpha\) \((0 < \alpha < \pi/2)\) and centre at the origin can be described as a rational Bézier curve \(^2\) of degree 2. Its control points are given by

\[
\begin{align*}
a_0 &= (r \cos \alpha, -r \sin \alpha) \\
a_1 &= \left( \frac{r}{\cos \alpha}, 0 \right) \\
a_2 &= (r \cos \alpha, r \sin \alpha)
\end{align*}
\]

and its weights are given by

\[
\begin{align*}
\hat{w}_0 &= 1 \\
\hat{w}_1 &= \cos \alpha \\
\hat{w}_2 &= 1
\end{align*}
\]

By elevating the degree \(^4\) of this rational Bézier curve, we produce the control points and weights of that cubic rational Bézier curve which describes the previous circular arc.

Denoting the new control points by \(b_i\) and the new weights by \(w_i\), we get

\[
\begin{align*}
w_i &= \hat{w}_{i-1} \frac{i}{3} + \hat{w}_i \left(1 - \frac{i}{3}\right) \\
\hat{w}_{i-1}^{-1} &= \frac{\hat{w}_i}{3} a_{i-1} + \hat{w}_i \left(1 - \frac{i}{3}\right) a_i \\
b_i &= \frac{\hat{w}_i}{w_i}
\end{align*}
\]

where

\[
\begin{align*}
w_0 &= 1 \\
w_1 &= \frac{1}{3}(1 + 2 \cos \alpha) \\
w_2 &= \frac{1}{3}(1 + 2 \cos \alpha) \\
w_3 &= 1
\end{align*}
\]

Using the notation \(w = w_1 = w_2\), we find that

\[
\begin{align*}
b_0 &= a_0 \\
b_1 &= \frac{a_0 + 2 \cos \alpha a_1}{3w} \\
b_2 &= \frac{a_2 + 2 \cos \alpha a_1}{3w} \\
b_3 &= a_2
\end{align*}
\]

The result is shown in Figure 1.

The next step is to lift out the control points from the \((x, y)\) coordinate plane along a line parallel to the \(z\) axis. The first and last control points, i.e. \(b_0\) and \(b_3\), are fitted onto the helix, that is

\[
\begin{align*}
h_{0z} &= -\alpha p \\
b_{3z} &= \alpha p
\end{align*}
\]

without loss of generality. Owing to this symmetry, \(b_{1z}\) and \(b_{2z}\) differ only in their sign. Thus we apply the notation

\[
\begin{align*}
b_{1z} &= -b \\
b_{2z} &= b
\end{align*}
\]

Now \(b\) is the only free parameter for the unique determination of the approximating twisted curve. The coordinates of the control points are finally

\[
\begin{align*}
b_0 &= (r \cos \alpha, -r \sin \alpha, -\alpha p) \\
b_1 &= \left( \frac{2 + \cos \alpha}{r + 2 \cos \alpha}, -r \frac{\sin \alpha}{1 + 2 \cos \alpha}, -b \right) \\
b_2 &= \left( \frac{2 + \cos \alpha}{r + 2 \cos \alpha}, r \frac{\sin \alpha}{1 + 2 \cos \alpha}, b \right) \\
b_3 &= (r \cos \alpha, r \sin \alpha, \alpha p)
\end{align*}
\]

Figure 1 Control points of cubic rational Bézier curve which describes circular arc under consideration.
The coordinate functions of the approximating cubic rational Bézier curve are

\[ r_x(t) = \frac{(2t^2 - 2t + 1) \cos \alpha + 2t(1 - t)}{2t - 1} \]
\[ r_y(t) = \frac{2t - 1}{2t(1 - t) \cos \alpha + 2t^2 - 2t + 1} \]
\[ r_z(t) = \frac{\alpha p(1 - t^2) + b(1 + 2 \cos \alpha)t(1 - t)}{2t(1 - t) \cos \alpha + 2t^2 - 2t + 1} \]

EXACT SLOPES AT END-POINTS

The free parameter \( b \) can be determined in several ways. One option is to determine \( b \) in such as to ensure exact slopes at the end-points, i.e. to enforce the equality of the angle between the tangent line of the rational Bézier curve and the \((x, y)\) coordinate plane, and the angle between the tangent line of the helix and the same coordinate plane at the first and last points of the arc. This means that the helix and the approximating curve have common tangent lines, but not necessarily common first derivatives, at the end-points.

Using the previously introduced notation, this requirement implies the equality

\[ \frac{p}{r} = \frac{b_{3z} - b_{2z}}{(b_{3z} - b_{2z})^2 + (b_{3y} - b_{2y})^2}^{1/2} \]

from which

\[ b = b_{2z} = p \left( \alpha \frac{2 \sin \alpha}{1 + 2 \cos \alpha} \right) \]  

This choice of \( b \) results in a cubic rational Bézier curve with the following properties:

- All of its points are on the cylinder of the helix.
- The points at the parameter values \( t = 0, t = 0.5 \) and \( t = 1 \) are on the helix.
- The tangent lines at the end-points coincide with the tangent lines of the helix.

The deviation of the approximating curve from the helix is of interest. A difference between the approximating rational Bézier curve and the helix can only exist in terms of their \( z \) coordinates. The arc of the helix is \( h(v), v \in [-\alpha, \alpha] \) (see Equation 1) and the approximating curve is \( r(t), t \in [0, 1] \) (\( b \) is substituted in Equations 4 with Equation 5). We apply a domain transformation to the approximating curve by changing \([0, 1]\) to \([-\alpha, \alpha]\), i.e. by the function \( t(u) = (u + 1)/2\alpha, u \in [-\alpha, \alpha] \). Then we have to find the matching \( u \) and \( v \) values, i.e. those which correspond to the same generator of the cylinder.

\[ \delta(\alpha, v) = r_z(u) - pv \]

This results in the equalities

\[ h_z(v) = b_z(u) \]
\[ r_z(v) = \frac{2\alpha u}{u^2(1 - \cos \alpha) + \alpha^2(1 + \cos \alpha)} \]

This yields a 2nd-degree polynomial in \( u \). From its two roots we choose

\[ u = \left( \frac{\alpha \sin \alpha}{1 - \cos \alpha} \right) \left( \frac{1 - \cos \alpha}{\sin \alpha} \right) \]  

by the preliminary conditions \( u \in [-\alpha, \alpha] \) and \( v \in [-\alpha, \alpha] \).

The error function is given by

\[ \delta(\alpha, v) = r_z(u) - pv \]

\[ \frac{\partial \delta}{\partial v} = 0 \]

This error surface is shown in Figure 2.

Because of symmetry, it is sufficient to examine the error on the domain \( \alpha \in (0, \pi/2), v \in [0, \alpha] \). At any fixed value of \( \alpha (\alpha = \hat{\alpha}) \), we can calculate the \( v \) that gives the maximum error. In order to find this \( v \), the equation

\[ \frac{\partial \delta(\alpha, v)}{\partial v} = 0 \]

has to be solved, which results in

\[ \cos v \cos^2 \alpha + 2 \cos^2 \alpha \alpha + \alpha \cos \alpha \sin \alpha - \cos v - 2 \]
\[ + \alpha \sin \alpha + \alpha \cos \alpha \sin \alpha = 0 \]
\[ 2 \sin \alpha - \alpha (1 + \cos \alpha) \]
\[ \frac{\alpha - \sin \alpha}{\alpha - \sin \alpha} \]  

This error surface is shown in Figure 2.

Figure 2 Error surface in case of exact slopes
Now we show that $\delta(\alpha, u)$ is monotone increasing in $\alpha$ on the specified domain. In order to prove this, it is sufficient to verify that the inequality
\[
(1 - \cos u)(\cos(\alpha - \cos u)(\alpha(1 + 2\cos \alpha)) - \sin \alpha(2 + \cos \alpha))
\]
holds. Since the denominator and the first factor of the numerator are positive, and the second factor of the numerator is nonpositive, it is enough to justify the inequality
\[
\alpha(1 + 2\cos \alpha) \leq \sin \alpha(2 + \cos \alpha)
\]
(8)

By Maclaurin’s theorem, the inequalities
\[
\begin{align*}
\sin \alpha & \geq \alpha - \frac{\alpha^3}{6} \\
\cos \alpha & \geq 1 - \frac{\alpha^2}{2} \\
\cos \alpha & \geq 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{24}
\end{align*}
\]
are valid. Decreasing the right-hand side and increasing the left-hand side of Expression 8, we get
\[
\alpha(1 + 2\cos \alpha) \leq 3\alpha - \alpha^3 + \frac{\alpha^5}{12} \leq \sin \alpha(2 + \cos \alpha)
\]
i.e. Expression 8 is justified.

A consequence of these properties of $\delta(\alpha, u)$ is that the maximum error is monotone decreasing to zero as $\alpha$ decreases to 0. This property of the maximum error enables us to choose an $\alpha$ which guarantees that the error does not exceed a prescribed tolerance.

Assume that we are given any tolerance $\epsilon > 0 \in \mathbb{R}$.
To find the maximum value of $\alpha$ which satisfies our requirements, we have to solve the equation $\delta(\alpha, u) = \epsilon$. This can be rearranged in the form $\delta(\alpha, u(\alpha)) = \epsilon/p$. In other words, the optimal value of $\alpha$ can be expressed as a function of $\epsilon/p$. Figure 3 shows a graph of this function.

C² CONTINUITY AT JUNCTIONS

In practice, it is quite common that the approximating curve has to be composed of several arcs. Considering our initial conditions, this is the case in which the helical arc to be approximated is greater than half of the pitch. In these cases, it is advisable to ensure a certain degree of continuity at junctions. In the method described above, there can only be 1st-order ($C^1$) continuity at junctions, because even the continuity of osculating planes is not guaranteed, which is a necessary condition for $C^2$ continuity.

Our first aim is to determine $b$ in order to ensure common osculating planes at junctions.

Let us consider the cubic rational Bézier curve $r(t)$, $t \in [0, 1]$, determined by the control points $b_0, b_1, b_2, b_3$ and the weights $w_0 = 1, w_1 = w_2 = w_3 = 1$, and the connecting cubic rational Bézier curve $r(t)$, $t \in [1, 2]$, $b_1 = b_0, b_1, b_2, b_3, \tilde{w}_0 = w_0, \tilde{w}_1 = w_1, \tilde{w}_2 = w_2, \tilde{w}_3 = w_3$ (see Figure 4). The curve $r(t)$ is obtained from $r(t)$ by a helical motion. This helical motion is determined by $\alpha, \beta$, and its axis is the $z$ axis, i.e. the axis of the helix to be approximated.

These two curves have a common osculating plane at the junction point $b_1 = b_0$ if the control points $b_1, b_2, b_3, b_4$ are coplanar. Owing to the symmetry, these control points are coplanar if the lines $b_1, b_2$ and $b_3, b_4$ are intersecting, and the distance from their point of intersection $m$ to the $(x, y)$ coordinate plane is equal to the distance from the control point $b_1 = b_0$ to the same coordinate plane.

Using the notation of Figure 5,
\[
\begin{align*}
\frac{ap}{b} &= \frac{m_x}{b_{2y}} \\
\frac{b - b_{2y}ap}{m_y}
\end{align*}
\]
from which
\[
\begin{align*}
b &= \frac{b_{2y}ap}{m_y}
\end{align*}
\]
Figure 3 Optimal values of $\alpha$ for exact slopes

Figure 4 Control points of two connecting Bézier curves

Figure 5
By the equality
\[ m_y - b_2, \tan \alpha = \frac{2r + r \cos \alpha \sin \alpha}{1 + 2 \cos \alpha \cos \alpha} \]
we get the expression
\[ b = ap \frac{\cos \alpha}{2 + \cos \alpha} \]

The resulting rational Bézier curves have a common osculating plane at the junction, i.e., at the parameter value \( t = 1 \). Let us examine whether there is a \( C^2 \) continuity. The condition for \( C^1 \) continuity is
\[ \frac{3w_2 (b_3 - b_2)}{w_3} = \frac{3w_1}{w_0} (b_1 - b_0) \]
(see Equations 2). This equality holds because \( w_0 = w_3 = 1, w_2 = w_1 = w \), and \( b_1 - b_2 = b_1 - b_0 \).

The further condition for \( C^2 \) continuity is the equality
\[ b_2 - 2 \cos \alpha (b_1 - b_0) = b_1 + 2 \cos \alpha (b_3 - b_2) \]
which can be derived from Equations 3. Owing to symmetry (see Figure 6), this equality holds if the equalities
\[ \ell_{22} - 2 \cos \alpha (\ell_{12} - \ell_{02}) = \ell_{02} \]
\[ \ell_{12} + 2 \cos \alpha (\ell_{32} - \ell_{22}) = \ell_{32} \]
are valid. This results in \( \cos \alpha = 1, \) i.e., there can be \( C^2 \) continuity only in the case in which \( \alpha = 0 \).

Neither can \( C^3 \) continuity be gained by the application of an affine parameter transformation, since, for \( C^1 \) continuity, the equality of Equations 2 requires equal-length parameter ranges under the previous circumstances.

Denoting the curvature of \( r(t) \) at \( t = 1 \) by \( \kappa \), and the curvature of \( \mathcal{R}(t) \) at \( t = 1 \) by \( \mathcal{K} \),
\[ \kappa = \frac{2w_2 w_1 (b_3 - b_2) \times (b_2 - b_1)}{3w_2^2 |b_3 - b_2|^3} \]
\[ \mathcal{K} = \frac{2w_0 w_2 (b_1 - b_0) \times (b_2 - b_0)}{3w_1^2 |b_1 - b_0|^3} \]
\( \kappa = \mathcal{K} \) since \( w_0 = w_3 = 1 \) and \( w_2 = w_1 = w \). This means that these curves not only have common osculating planes but also equal curvatures at the junction.

Summarizing our results, we can see that the approximating cubic rational Bézier curve has the following properties:
- All of its points are on the cylinder of the helix.
- The points at the parameter values \( t = 0, t = 0.5 \) and \( t = 1 \) are on the helix.
- There is a 2nd-order (\( G^2 \)) geometric continuity at the junction.

In order to achieve \( C^2 \) continuity, we carry out a projective parameter transformation which, as is already known, results in the transformation of weights. Thereby, the rational Bézier curve determined by the control points \( b_i \) and weights \( w_i (i = 0, 1, \ldots, n) \) and the curve determined by the control points \( b_i \) and weights \( c w_i (i = 0, 1, \ldots, n) \), \( 0 < c \in \mathbb{R} \), are two different representations of the same curve. Different points belong to the same parameter value in different representations.

We apply a projective parameter transformation for the cubic rational Bézier curves of Figure 4. Consider the cubic rational Bézier curve \( r(t), t \in [0, 1] \), determined by the control points \( b_0, b_1, b_2, b_3 \) and the weights \( w_0 = 1/c^3, w_1 = w/c^2, w_2 = w/c, w_3 = 1 \), and the curve \( \mathcal{R}(t), t \in [1, 2] \), of the same type determined by the control points \( b_0, b_1, b_2, b_3 \) and the weights \( w_0 = 1, w_1 = c^2, w_2 = c^2w, w_3 = c^3 \). We examine whether there are positive constants \( c \) and \( \hat{c} \) which guarantee \( C^2 \) continuity of the curves \( r(t) \) and \( \mathcal{R}(t) \) at the parameter value \( t = 1 \).

For \( C^0 \) continuity, \( b_3 = \bar{b}_0 \) must hold. For \( C^1 \) continuity,
\[ \frac{3w_2}{w_3} (b_3 - b_2) = \frac{3w_1}{w_0} (b_1 - b_0) \]
This results in \( 1/c = \hat{c} \), since \( b_3 - b_2 = \bar{b}_1 - \bar{b}_0 \). Making use of this, we get the weights \( w_0 = c^3, w_1 = c^2w, w_2 = cw, w_3 = 1 \), and \( \bar{w}_0 = 1, \bar{w}_1 = cw, \bar{w}_2 = c^2w, \bar{w}_3 = c^3 \).

For \( C^2 \) continuity,
\[ \frac{3w_2}{w_3} (b_3 - b_2) - \frac{w_1}{w_3} (b_3 - b_1) \]
\[ = \frac{w_1 \bar{w}_0 - 3w_2^2}{w_0^2} (b_1 - \bar{b}_0) + \frac{w_2}{w_0} (b_2 - \bar{b}_0) \]
After some simplification, we get
\[
\frac{1}{c} = 3w - \frac{[b_2 - b_1]}{2[b_3 - b_2]}
\]

Considering the similar triangles of Figure 7, we find that
\[
c = \frac{1}{\cos \alpha}
\]

By this choice of \(c\), a \(C^2\) continuity is ensured at the junction.

The error calculations are analogous to that of the previous section. We apply the same domain transformation, and we get the error function
\[
\delta (\alpha, v) = \rho \left( \frac{Au^2 + Bu}{Cu^2 + D} - v \right)
\]
\[
\alpha \in (0, \pi/2); \quad v \in [-\alpha, \alpha]
\]

where
\[
A = 1 - \cos^3 \alpha
\]
\[
B = \alpha^2(3 + 2 \cos \alpha + \cos^2 \alpha)
\]
\[
C = 2 - \cos \alpha - \cos^2 \alpha
\]
\[
D = \alpha^2(2 + \cos \alpha + \cos^2 \alpha)
\]

and \(u\) is determined by Equation 6.

This error function has the same properties as the function in Equation 7. Owing to the symmetry, it is enough to examine the error on the domain \(\alpha \in (0, \pi/2), v \in [0, \alpha]\).

At any fixed value of \(\alpha(\alpha - \tilde{\alpha})\), we can compute the \(v\) that gives the maximum error. For this, we have to find a root of \(\delta (\tilde{\alpha}, v)/\partial v = 0\). This means finding a root of
\[
\cos^3 v(E^3 C^2 - 2E^2 CD + D^2) + \cos^2 v(-E^4 C^2 - 2E^2 CD + 3D^2 - E^3 AC + E^3 (3AD - BC) - EBD) + \cos v(-E^4 C^2 + 2E^2 CD + D^2 - E^3 AC) - E^3 (3AD - BC) - EBD = 0
\]

The error function increases monotonically in \(\alpha\) on the specified domain, since the inequality
\[
\frac{\partial \delta (\alpha, v)}{\partial \alpha} = \frac{(1 - \cos v)}{(\cos^2 \alpha - 1)\cos \alpha + 2)^2 \sin v}
\]
\[
(\alpha (5 \cos^2 \alpha + 2 \cos v \cos^2 \alpha + 2 \cos \alpha \cos v - \cos v + 8 \cos \alpha + 2) - \sin \alpha (\cos \alpha \cos v + 2 \cos v + 10 \cos \alpha + \cos^3 \alpha + 4)) \geq 0
\]
can be proved. To show this, we must justify the inequality.
\[
\alpha (5 \cos^2 \alpha + 2 \cos v \cos^2 \alpha + 2 \cos \alpha \cos v - \cos v + 8 \cos \alpha + 2) \leq -\sin \alpha (\cos \alpha \cos v + 2 \cos v + 10 \cos \alpha + \cos^3 \alpha + 4)
\]

This can be proved in a similar way to Expression 8, i.e. one can use the inequalities derived from Maclaurin's theorem (but with more terms) and the inequality cos \(v \geq \cos \alpha\). A consequence of these properties of \(\delta (\alpha, v)\) is that the maximum error decreases monotonically to zero as \(\alpha\) decreases to 0.

Making use of this property of the maximum error, the optimal \(\alpha\) can be determined for any prescribed error limit, as in the previous section. However, in this case, the evaluation of \(\delta (\tilde{\alpha}, v)\), i.e. the computation of the maximum value of \(\delta\) for a fixed \(\alpha = \tilde{\alpha}\), is more complicated. The optimal value of \(\alpha\) can be considered as a function of \(\epsilon/p\), i.e. \(\delta (\alpha, v(\alpha)) = \epsilon/p\). Figure 9 shows a graph of this function.

**CONCLUDING REMARKS**

In the previous section, we have shown that two approximating segments can be joined to form a \(C^2\) curve. Unfortunately, it is impossible to join more than
Approximating the helix with rational cubic Bézier curves: I Juhász

degree of continuity at junctions decreases the accuracy of the approximation, i.e. for the same accuracy, more approximating arcs have to be applied.

Since the error of the approximation is proportional to $p$, the increase of $p$ decreases the optimal $\alpha$ (see Figures 3 and 9), i.e. for a very large $p$, $\alpha$ is too small to be applicable in practice. However, in practical applications such as screw threads, $p$ is small.

By means of the method described above, helicoids can be approximated by rational Bézier patches. If the helicoid is generated by a helical motion with parameter $p$ of a rational Bézier curve, then the vertical error of this approximation is equal to the error of the approximation of a helix of the same parameter. Figure 11 shows an axonometric view of a helicoid approximated by rational Bézier patches.

**ACKNOWLEDGMENTS**

The author would like to thank Z. S. Bancsi for discussions with him, and the referees of this paper for their helpful comments and suggestions.

**REFERENCES**


Imre Juhász is a lecturer at the Department of Descriptive Geometry at the University of Miskolc in Hungary. He graduated from Kossuth Lajos University of Sciences and Arts in Debrecen, from where he also obtained his doctorate. His research interests are in geometric modelling, especially the description of curves and surfaces, algorithms for computer graphics, and graphics standards.